GENERALISED NON-ORTHOGONAL DESIGN AND ITS ANALYSIS WITH RECOVERY OF INTERBLOCK INFORMATION

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1. Introduction

After the Generalised Balanced Designs were introduced by Das (1957), a Generalised Partially Balanced Design was defined and discussed by Raut and Das (1959). The randomised block, P.B.I.B. and reinforced P.B.I.B. designs come out as particular cases of the Generalised Partially Balanced Designs. The method of intrablock analysis of this design was included in the paper by Raut and Das. In experiments involving single plant progenies with variable amount of seeds or small number of treatments with litter mates as experimental units, it becomes desirable to adopt super-complete block designs, *i.e.*, designs allowing replication within groups (blocks) in order to avoid wastage of seed or animals. The generalised designs obtained so far are generalisation of only two incomplete block designs, *viz.*, B.I.B. and P.B.I.B. There are other types of incomplete block designs also for which no generalisation has yet been attempted. In the present paper an attempt has been made to evolve a method of both intra and interblock analysis of any design generalised as in the case of the two designs mentioned earlier.

Though there is a general method of analysis with recovery of interblock information in the case of incomplete block designs with cell frequencies '0' and '1', the same method cannot be directly extended for obtaining combined estimates when the cell frequencies in the design are other than '0' and '1'. In the present paper an attempt has been made to present general methods of intra and inter-intrablock combined analyses when the design is obtained by generalization of any of the incomplete block designs.

2. Generalised Non-orthogonal Design

A generalised non-orthogonal design may be defined as follows:

Let there be $v' + v\ (= t)$ treatments divided into two sets, one containing v' treatments and the second v treatments and 'b' blocks such that any treatment t_m belonging to the first set of v' treatments ($m = 1, 2, \ldots v'$) occurs in each of the 'b' blocks $n\ (\geqslant 0)$ times, while any treatment t_i belonging to the second set of v treatments ($i = v' + 1, v' + 2, \ldots, v' + v$) occurs in the '$b$' blocks either S or $S + p$ times where $S \geqslant 0$, $p \geqslant 0$, such that the cells receiving frequency $S + p$ form a non-orthogonal design with v treatments and 'b' blocks.

To these 'b' blocks another set of b' blocks may be added such that in any of these blocks, each of the treatments occurs the same number of times, say f_i where f_i may change from block to block. This design with $v' + v\ (= t)$ treatments in $b + b'$ blocks has been defined to be the generalised non-orthogonal design.

When written in tabular form the design looks like that given in the previous paper (Raut and Das, 1959). For the facility of defining the different parameters involved in the design and explaining the other symbols used in connection with the analysis, the design in form of the cell frequencies has been reproduced on next page.

where

$$R_1 = bn + \sum_{i=1}^{b'} f_i = bn + \Sigma f$$

$$R_2 = bS + rp + \sum_{i=1}^{b'} f_i = R + \Sigma f,$$

where

$$R = bS + rp$$

$$T = v'R_1 + vR_2 = bK + t\,\Sigma f.$$

3. Intrablock Analysis of the Design

For the intrablock estimate of the treatment effects, the model considered is as usual,

$$Y_{ijk} = \mu + t_i + b_j + \epsilon_{ijk}$$

where

Y_{ijk} : the k-th observation from the i-th treatment in the j-th block;

μ : the common contribution (a constant);

t_i : effect of the i-th treatment;

b_j : the contribution of block j;

ϵ_{ijk} : random variable with mean 'zero' and variance, σ^2.

Table of frequencies

Blocks	First set of treatments					Second set of treatments				Totals
	1	2	3	...	v'	$v'+1$	$v'+2$	...	$v'+v$	
1	n	n	n	...	n	$S+p$	S	...	S	$K=nv'+vS+kp$
2	n	n	n	...	n	S	S	...	$S+p$	K
3	n	n	n	...	n	S	$S+p$	...	S	K
.	.	.	.	...	.	.	.	...	.	.
.	.	.	.	...	.	.	.	...	.	.
.	.	.	.	...	.	.	.	...	.	.
b	n	n	n	...	n	S	$S+p$	...	$S+p$	K
$b+1$	f_1	f_1	f_1	...	f_1	f_1	f_1	...	f_1	tf_1
$b+2$	f_2	f_2	f_2	...	f_2	f_2	f_2	...	f_2	tf_2
.	.	.	.	...	.	.	.	...	.	.
.	.	.	.	...	.	.	.	...	.	.
.	.	.	.	...	.	.	.	...	.	.
$b+b'$	$f_{b'}$	$f_{b'}$	$f_{b'}$	...	$f_{b'}$	$f_{b'}$	$f_{b'}$	...	$f_{b'}$	$tf_{b'}$
Totals ..	R_1	R_1	R_1	...	R_1	R_2	R_2	...	R_2	T

The normal equations for the estimation of the treatment effects after eliminating the block effects have been shown below when one of the treatments in the first set is eliminated using the restriction $\sum_{i=1}^{v'+v} t_i = 0$. The normal equations for the first set of treatments come out as

$$R_1 t_m - \frac{n(R - bn)}{K} \sum_{i=v'+1}^{v'+v} t_i = Q_m \tag{1}$$

where Q_m is the adjusted total of the m-th treatment and is given by

$$Q_m = T_m - \frac{n}{K}\sum_{j=1}^{b} B_j - \frac{1}{t}\sum_{j'=b+1}^{b+b'} B_j' \quad (m = 1, 2, \ldots v', \text{ the treatments in the first set having the constant cell frequency in all the blocks.}$$

Adding over all the v' treatments in the first set and using the fact that $\sum_{i=1}^{v'+v} t_i = 0$, we can get;

$$\sum_{i=v'+1}^{v'+v} t_i = \frac{-K\Sigma Q_m}{Rnt + K\Sigma f}. \tag{2}$$

Substituting the value of Σt_i from (2) in (1) we get

$$t_m = \frac{1}{R_1}\left[Q_m - \frac{n(R - bn)\Sigma Q_m}{Rnt + K\Sigma f}\right]. \tag{3}$$

This expression is the same as obtained before in case of Generalised Partially Balanced Design defined in the previous paper by the author (1959).

The normal equations for the second set of treatments are

$$\left(R_2 - \frac{P}{K}\right) t_i - \frac{1}{K}\sum_{i \neq i'} \{bS(S-n) + pr(2S-n) + p^2\lambda_{ii'}\}\, t_i' = Q_i \tag{4}$$

where

$$Q_i = T_i - \sum_{j=1}^{b+b'} \frac{n_{ij}B_j}{K_j},$$ n_{ij} being the frequency of the i-th treatment in the j-th block; $(i = v' + 1, v' + 2, \ldots, v' + v)$

and

$$P = r(S + p)(S + p - n) + (b - r)S(S - n).$$

Equation (4) can also be written in the form

$$\left(R_2 - \frac{rp^2}{K}\right) t_i - \frac{p^2}{K}\sum_{i \neq i'} \lambda_{ii'} t_{i'} = Q_i + \frac{A}{K}\sum t_i \tag{5}$$

where

$$A = bS(S - n) + pr(2S - n).$$

Substituting the value of Σt_i as given in (2), equation (5) reduces to

$$\left(R_2 - \frac{rp^2}{K}\right) t_i - \frac{p^2}{K} \sum_{i \neq i'} \lambda_{ii'} t_i' = Q_i + \frac{A \Sigma Q_i}{Rnt + K \Sigma f}. \tag{6}$$

It will be seen that equation (6) is exactly of the same form as would be obtained in the case of the incomplete block design which has been generalised. As such these equations for obtaining t_i can also be solved through the same technique as applied for the incomplete block design.

4. Intra-interblock Combined Analysis

For estimating the treatments from the block totals, the following model can be used:

$$B_j = K\mu + \Sigma n_{ij} \tau_i + Kb_j.$$

The equations for estimating μ, t_m and t_i are as follows:

(*i*) *For the first set:* Equation for any treatment t_m in the first set is:

$$G = \sum_j B_j = bK\mu + bn \, \Sigma \, t_m + R \Sigma t_i$$

or

$$G = bK\mu + (R - bn) \Sigma t_i$$

$$= bK\mu + (bn - R) \Sigma t_m. \tag{7}$$

(*ii*) *For the second set:* Equation for any treatment t_i in the second set is:

$$T_i = RK\mu + Rn \Sigma t_m + (bS^2 + 2rSp) \Sigma t_i + rp^2 t_i + p^2 \sum_{i \neq i'} \lambda_{ii'} t_{i'}. \tag{8}$$

Using the relation (7) the above equation reduces to

$$rp^2 t_i + p^2 \sum_{i \neq i'} \lambda_{ii'} t_{i'} = T_i - \frac{RG}{b} + \frac{r^2 p^2}{b} \sum t_i \tag{9}$$

From the equations (7) for the first set, it can be seen that no individual treatments in this set admits of any estimate from the block totals. Hence such treatments will have no interblock estimates.

A combined estimate of t_i can now be obtained by giving weights W to equation (6) and W'/K to equation (9),

where

$$W = \frac{1}{\sigma^2}$$

and

$$W' = \frac{1}{\sigma^2 + K\sigma_b^2}.$$

With the above weights the combined estimate comes out as

$$\left[WR_2 - \frac{rp^2}{K}(W - W')\right] t_i - \frac{p^2}{K}(W - W') \sum_{i \neq i'} \lambda_{ii'} t_i$$

$$= WQ_i + \frac{W'}{K}\left(T_i - \frac{RG}{b}\right) + \frac{WA\,\Sigma\, Q_i}{Rnt + K\Sigma f}$$

$$+ \frac{W'}{K} \cdot \frac{r^2p^2}{b} \sum t_{i\cdot} \tag{10}$$

Dividing both sides by $W - W'$, we get

$$Ct_i - \frac{p^2}{K} \sum_{i \neq i'} \lambda_{ii'} t_{i'} = P_{i'} \tag{11}$$

where

$$C = \frac{WR_2}{W - W'} - \frac{rp^2}{K}$$

and

$$P_{i'} = \frac{1}{(W - W')} \left\{ WQ_i + \frac{W'}{K}\left(T_i - \frac{RG}{b}\right) \right.$$

$$\left. + \frac{WA\,\Sigma\, Q_i}{Rnt + K\Sigma f} + \frac{W'}{K} \cdot \frac{r^2p^2}{b} \sum t_{i\cdot} \right\}$$

It will be seen that in this form the equation is exactly of the type as in the case of intrablock analysis. Though treatments in the first set have no combined estimate, the difference between two treatments, one belonging to the first set and the second to the other set, can be estimated from their difference, as for both the estimates the same restriction $\sum_{i=1}^{v'+v} t_i = 0$ has been used. Had two different restrictions been used for the two estimates, it would not have been possible to estimate such differences.

5. The Estimation of the Weights

We have taken

$$W = \frac{1}{\sigma^2}$$

and

$$W' = \frac{1}{\sigma^2 + K\sigma_b^2}.$$

Denoting the intrablock error mean square by E, we get the estimate, w of W to be

$$w = \frac{1}{E} \tag{12}$$

and denoting the block sum of square adjusted for treatment effects by B' which can be obtained from the relation

Adjusted treatment $S.S.$—Unadjusted treatment $S.S.$ = Adjusted block $S.S.$—Unadjusted block $S.S.$

it can be shown that

$$E(B') = b\left(K - \frac{k}{R}\right)\sigma_b^2 + (b-1)\sigma^2$$

or

$$\sigma_b^2 = \frac{B' - (b-1)E}{b\left(K - \frac{k}{R}\right)} \tag{13}$$

$$\therefore \quad w' = \frac{b(KR - k)}{KR(B' + E) - bkE} \tag{14}$$

which can be put in the alternative form,

$$w' = \frac{b(KR - k)}{KRB' + (KR - bk)E}. \tag{15}$$

6. Some Particular Cases

(*i*) When $p = 1$ and $S = n = f_i = 0$, the design becomes the usual partially balanced incomplete block design. Putting these values we get

$$\left.\begin{aligned} &K = k, \quad R_2 = R = r, \\ &w' = \frac{v(r-1)}{KB' - (v-k)E} \\ &\text{and} \\ &E(B') = v(r-1)\sigma_b^2 + (b-1)\sigma^2 \end{aligned}\right\}. \tag{16}$$

These are also the values of w' and $E(B')$ for the analysis of the ordinary P.B.I.B. design.

The combined estimate given at (11) reduces in this case to

$$C_1 t_i - \frac{1}{k}\sum_{i \neq i'} \lambda_{ii'} t_{i'} = P'_{i'} \tag{17}$$

where

$$C_1 = r\left[\frac{W}{W - W'} - \frac{1}{k}\right]$$

and

$$P'_{i'} = \frac{1}{(W - W')}\left\{WQ_i + \frac{W'}{k}\left(T_i - \frac{rG}{b}\right)\right\}.$$

(*ii*) In the case of balanced incomplete block designs, in addition to the relations given at (16), $\lambda_{ii'}$ is a constant, say, equal to λ and the expression (17) reduces to the simple form

$$\left(C_1 - \frac{\lambda}{k}\right) t_i = P'_{i'}. \tag{18}$$

(*iii*) When $n = p = 1$ and $f_i = S = 0$, the design becomes a reinforced P.B.I.B. design.

$$\left.\begin{aligned} &\text{Here} \\ &K = v' + k, \quad R_2 = R = r \\ &w' = \frac{b\left(r - \dfrac{k}{v' + k}\right)}{rB' + \left(r - \dfrac{bk}{v' + k}\right)E} \\ &\text{and} \\ &E(B') = \{bv' + v(r-1)\}\sigma_b^2 + (b-1)\sigma^2 \end{aligned}\right\}. \tag{19}$$

The combined estimate for the reinforced P.B.I.B. design is

$$C_2 t_i - \frac{1}{v' + k} \sum_{i \neq i'} \lambda_{ii'} t_{i'} = P''_{i'} \tag{20}$$

where

$$C_2 = r \left[\frac{W}{W - W'} - \frac{1}{v' + k} \right]$$

and

$$P''_{i'} = \frac{1}{W - W'} \left[W Q_i + \frac{W'}{v' + k} \left(T_i - \frac{rG}{b} \right) + \frac{\Sigma Q_i}{v' + v} \left(\frac{k}{v} W' - W \right) \right].$$

7. Summary

A type of design called generalised non-orthogonal design has been defined. A method of intrablock analysis has been described together with evolving a method for obtaining estimates of treatments using the block totals. Most of the existing incomplete block designs come out as particular cases of this design.

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